

# A Sufficient Condition for the Existence of a Principal Eigenvalue for Nonlocal Diffusion Equations with Applications.

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## Abstract

Considerable work has gone into studying the properties of non-local diffusion equations. The existence of a principal eigenvalue has been a significant portion of this work. While there are good results for the existence of a principal eigenvalue equations on a bounded domain, few results exist for unbounded domains. On bounded domains, the Krein-Rutman theorem on Banach spaces is a common tool for showing existence. This article shows that generalized Krein-Rutman can be used on unbounded domains and that the theory of positive operators can serve as a powerful tool in the analysis of nonlocal diffusion equations. In particular, a useful sufficient condition for the existence of a principal eigenvalue is given.

## 1 Introduction

Nonlocal diffusion equations come up in a number of contexts. These range from biology [1, 7, 10, 15], to materials science [3], and graph theory. Much of the analysis to this point has been done on bounded domains [4]. The results presented here represent a first step in analyzing nonlocal diffusion equations on unbounded domains. The theory of positive operators on Banach lattices is employed to give useful conditions to show existence of a principal

eigenvalue in the absence of compact domains. Many more results are likely to come from applying known results about positive operators to nonlocal diffusion equations. In particular, in Section 3.1, we use the existence of a particular type of maximum principal implies exponential convergence to zero.

The existence of a principal eigenvalue has been used to study many nonlocal diffusion equations, both linear and nonlinear [1, 6]. Energy methods and Fourier analysis have been used to provide polynomial bounds on the decay of nonlocal diffusion equations [2, 8], with exponential decay shown in some special cases [8]. However, estimates of the principal eigenvalue would be able to definitely show exponential decay. While much work has been done on the existence of principal eigenvalues on bounded domains, few general results exist for unbounded domains [4, 9]. It is important to note that, even on bounded domains, existence of a principal eigenvalue is still not guaranteed. See, for example, the counter example in section 3.2. The work here uses the theory of positive operators to find a sufficient condition for the existence of a principal eigenvalue without any assumptions on the boundary conditions. The method also gives a technique for estimating the value of the eigenvalue. Further work is needed to characterize the multiplicity of the eigenvalue or the existence of a spectral gap.

The general equation of interest is:

$$\dot{u}(x, t) = \int_{\Omega} J(x, y)u(y, t) \, dy - a(x)u(x, t) \quad (1.1)$$

where  $\Omega$  is some connected, possibly unbounded subset of the real line  $\mathbb{R}$  and  $u \in L^p(\Omega)$  for  $1 \leq p < \infty$ . This equation is often interpreted as modeling some form population dispersal. Individuals propagate from point  $y$  to point  $x$  at a rate  $J(x, y)$  and individuals die off at a rate  $a(x)$  depending on their location. Note that the proofs here immediately generalizes to  $\mathbb{R}^n$ .  $a$  is assumed to be continuous and bounded in the sense that there exists  $c, c'$  such that:

$$0 < c < a(x) < c' \quad (1.2)$$

The two additional hypotheses on  $J$  are that  $J(x, y) \geq 0$  and  $J$  is bounded with non-zero spectrum, where  $J$ :

$$Ju = \int_{\Omega} J(x, y)u(y) \, dy \quad (1.3)$$

The result in Theorem 1 gives a quite general condition for the existence of a principal eigenvalue for  $L$ . There is no particular reason that  $J$  has to be of integral type, but the discussion here will be limited to that case. If  $J$  is a more general positive operator, an appropriate compact topological space will have to be found to ensure the conclusion from Lemma 2. Several applications of the theorem are given in section 3.

Define the two related linear operators:

$$Lu = Ju - au \quad A_\lambda u = \frac{Ju}{\lambda + a} \quad (1.4)$$

$L$  is simply the operator that defines the dynamics of (1.1), and  $A_\lambda$  is a related family of operators. It is obvious that  $A_\lambda$  is positive whenever  $\lambda > -\inf a(x)$ . The two operators are related in the sense that  $L$  has  $\lambda > -\inf a$  as an eigenvalue if and only if  $A_\lambda$  has an eigenvalue equal to one.

The theorem can be stated as such:

**Theorem 1.** *If:*

$$\lim_{\lambda \rightarrow (-\inf a)^+} \text{spr}(A_\lambda) > 1 \quad (1.5)$$

where  $\text{spr}(A_\lambda)$  is the spectral radius and one of the conditions in Lemma 2 holds, then  $L$  has a principal eigenvalue,  $\lambda_0 \in \mathbb{R}$ , with positive eigenfunction such that for any element  $\lambda \in \sigma(L)$ , we have the inequality  $\Re \lambda \leq \lambda_0$ .

The condition in Theorem 1 holds for any boundary conditions and is not dependent upon the domain being bounded.

A quick lemma that gives a few conditions necessary for the existence of an eigenvalue:

**Lemma 2.**  $A_\lambda$  has a positive eigenvalue equal to its spectral radius with a positive eigenfunction in  $L^1$  whenever  $\lambda > -\inf a$  if any of the following conditions hold:

1. Range of  $J^n \subset L^p$  for  $1 < p < \infty$  for some  $n \in \mathbb{N}$ .
2.  $J(x, y) = K(x - y)$  is of convolution type with  $K$  absolutely bounded.
3.  $\exists c(x) \in L^1$  such that  $J(x, y) \leq c(x)$

*Proof.*  $A_\lambda$  is obviously a positive operator on the appropriate lattice in the sense that it preserves the positive cone. Because of the boundedness of  $a + \lambda$ , we know that  $A_\lambda$  has range in the same function space(s) as  $J$ . For that reason, the rest of the proof will only be concerned with  $J$ . Also, recall that we are assuming  $J$  has non-zero spectrum for the entirety of this article.

For condition 1, we know that all of the  $L^p$  spaces where  $1 < p < \infty$  are weakly compact, so we can invoke the generalized Krein-Rutman theorem on locally convex topological spaces immediately since  $A_\lambda$  has range in a compact topological space. [14]

Condition 2 allows us to consider the case where  $J$  is from  $L^1$  to  $L^1$ . Recall that the unit ball of  $ca(\Sigma)$ , the space of countably additive measures with respect to a  $\sigma$ -algebra  $\Sigma$ , is compact with respect to the weak topology, and we can again use Krein-Rutman to get the existence of an eigenvalue. Then, for any eigenfunction  $u$ , we can write:

$$Ju = (\lambda + \kappa)u \quad (1.6)$$

It is easy to show that  $u$  must be absolutely continuous with respect to the Lebesgue measure because of the convolution with a bounded function. Therefore,  $u$  must be isometric to an  $L^1$  function by the Radon-Nikodym theorem.

Finally, condition 3 allows us to make the same argument as for condition 2. It is straightforward to show that  $J : ca(\Sigma) \rightarrow L^1$  and that any eigenfunction must be an  $L^1$  function.  $\square$

The three conditions presented in Lemma 2 are surely not exhaustive, as will be seen in Section 3. More generally, if  $J$  is a bounded operator from  $ca(\Sigma) \rightarrow L^1$ , the conclusions also hold.

We need one more lemma about  $A_\lambda$ .

**Lemma 3.**  *$\text{spr}(A_\lambda)$  is a continuous, monotonically decreasing function of  $\lambda$  which converges to 0.*

*Proof.* First, we want to show that  $\text{spr}(A_\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . To show that, simply observe that we have the  $L^1$  operator norm inequality:

$$\|A_\lambda\|_{op} \leq \left\| \frac{1}{\lambda + a} \right\|_\infty \|J \star\|_{op} = \frac{1}{\lambda + \inf a} \quad (1.7)$$

The right hand side of that equation implies the norm of  $A_\lambda$  goes to zero asymptotically, which bounds the spectral radius above. Finally, we want to show that the spectral radius of  $A_\lambda$  is a monotonically decreasing, continuous function of  $\lambda$ . From previous results, we know that  $\text{spr}(A_\lambda)$  is upper-semicontinuous [12], and that  $A_\lambda \leq A_\mu$  implies that  $\text{spr}(A_\lambda) \leq \text{spr}(A_\mu)$  [11]. Since  $A_\lambda < A_\mu$  whenever  $\lambda > \mu$ , we have that  $\text{spr}(A_\lambda)$  is a monotonically decreasing, upper-continuous function.

It remains to show that the spectral radius function is lower-continuous. Define the function  $c(\lambda, \lambda')$  for  $\lambda, \lambda' > -\inf a$ :

$$c(\lambda, \lambda') = \sup_{x \in \Omega} \frac{\lambda + a(x)}{\lambda' + a(x)} \quad (1.8)$$

Observe that  $c$  is a continuous function and  $c(\lambda, \lambda) = 1$  for all  $\lambda > -\inf a(x)$ . Assume  $\lambda > \lambda'$ . Then,  $c(\lambda, \lambda') > 1$ . We also have the two inequalities:

$$A_\lambda \leq A_{\lambda'} \quad c(\lambda, \lambda') A_\lambda \geq A_{\lambda'} \quad (1.9)$$

We can now directly calculate:

$$\lim_{\lambda \downarrow \lambda'} \text{spr}(A_\lambda) \geq \lim_{\lambda \downarrow \lambda'} c(\lambda, \lambda') \text{spr } A_{\lambda'} = \text{spr}(A_{\lambda'}). \quad (1.10)$$

which completes the proof of lower-continuity.  $\square$

Note that the result of the above lemma holds even without any of the necessary conditions for Lemma 2. That fact becomes important for the result in Section 3.1. We can now move on to the rest of the proof.

## 2 Proof of the Theorem

First, we will show that the operator  $L$  is positive resolvent. Choose  $\lambda \in \mathbb{R}$  so large that  $\mu \geq \lambda$  implies  $\mu \in \rho(L)$  the resolvent of  $L$  and that  $\text{spr}(A_\mu) < 1$ . Assume  $f \geq 0$ ,  $f \in L^1$ , is in the range of  $\mu - L$ :

$$\mu v - Lv = f \quad (2.1)$$

for some  $v \in L^1$ . Expanding the above equation gives:

$$\begin{aligned}
\mu v - Jv + av &= f && \Longleftrightarrow \\
v - \frac{Jv}{\mu + a} &= \frac{f}{\mu + a} && \Longleftrightarrow \\
v - A_\lambda v &= \frac{f}{\mu + a} && \Longleftrightarrow \\
v &= \sum_{j=0}^{\infty} A_\lambda^j \frac{f}{\mu + a} \geq 0 && (2.2)
\end{aligned}$$

Since we have assumed that  $\text{spr}(A_\mu) < 1$ , the above series converges and  $(\mu - L)^{-1} \geq 0$ . By [13], we know that there exists  $\lambda_0$  such that:

$$\text{spr}\left((\mu - L)^{-1}\right) = \frac{1}{\mu - \lambda_0} \quad (2.3)$$

for all  $\mu > \lambda_0$ . We also have the characterization:

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \mid (\lambda - L)^{-1} \geq 0\} \quad (2.4)$$

Assume  $\lambda_0 > -\inf a$ . By [14], we know that:

$$(I - A_\lambda)^{-1} \geq 0 \iff \text{spr}(A_\lambda) < 1 \quad (2.5)$$

The above condition implies that  $\text{spr}(A_\lambda) < 1$  for all  $\lambda > \lambda_0$  and  $\text{spr}(A_{\lambda_0}) \geq 1$ . The continuity of  $\text{spr}(A_\lambda)$  implies that  $\text{spr}(A_{\lambda_0}) = 1$ . From our application of the Krein-Rutman theorem, we know there exists  $u \geq 0$  such that:

$$A_{\lambda_0} u = u \quad (2.6)$$

From our definition of  $A_\lambda$ , the above implies:

$$Lu = \lambda_0 u \quad (2.7)$$

$\lambda_0$  is therefore our principal eigenvalue and  $u$  its associated positive eigenfunction.

All that remains to be shown is that  $\lambda_0 > -\inf a$ . Recall the hypothesis on the spectral radius limit:

$$\lim_{\lambda \rightarrow (-\inf a)^+} \text{spr}(A_\lambda) > 1 \quad (2.8)$$

That inequality implies there exists a largest  $\lambda_0 > -\inf a$  such that  $\text{spr}(A_{\lambda_0}) = 1$ , and we are finished.

Note that in the proof of the theorem, the structure of the integral equation only comes up in Lemma 2. Define the more general operator  $B$  as:

$$Bf = Kf - af \quad (2.9)$$

where  $K$  is some positive operator. The only conditions we need for Theorem 1 to hold are that  $K$  has non-zero spectrum and the range of  $K^n$  is inside a compact topological space for some finite  $n \in \mathbb{N}$ .<sup>1</sup> In particular, if  $K^n \rightarrow U \subset L^p$  for  $1 < p < \infty$ , Theorem 1 holds.

### 3 Applying the Theorem

The result in Theorem 1 can be directly applied to prove the existence of a principal eigenvalue for a variety of problems. First, we will show that the existence of a maximum principal implies exponential convergence even without the existence of a principal eigenvalue. Then, we will give a couple of abstract propositions with and without symmetry conditions on  $J$ .

#### 3.1 Maximum Principle

We define our maximum principle similar to [4] as the condition:

$$Lu \leq 0 \implies u \geq 0 \quad (3.1)$$

Recall that we can rewrite the above equation to read:

$$Lu = f \leq 0 \iff (I - A_0)u = g \geq 0 \quad (3.2)$$

where  $-a(x)g(x) = f(x)$ . Our maximum principle is now equivalent to the assertion that  $(I - A_0)^{-1}$  is a positive operator. Using a result from Appendix 2 of [14], we know that  $(I - A_0)^{-1}$  is positive if and only if  $\text{spr}(A_0) < 1$ . Recalling the proof of Theorem 1, we know that  $\text{spr}(A) < 1$  if and only if  $\text{spr}(L) < 0$ . Our maximum principle therefore implies exponential convergence to zero for the nonlocal diffusion equation defined by  $L$ .

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<sup>1</sup>See Theorem 6.6 in Appendix V. and the following example in [14] for details.

### 3.2 Existence Propositions

For all of the results in this section, we will assume at least one of the conditions in Lemma 2 holds.

**Proposition 4.** *Assume  $a(x)$  is locally Lipschitz and reaches its infimum at  $x^*$  and  $J(x, y) = J(y, x)$ . Also, assume the range of  $J$  is in  $L^2$  and there exists an open neighborhood  $U \subset \mathbb{R}^2$  of  $(x^*, x^*)$  such that  $J(x, y) > c$  for all  $(x, y) \in U$ . Then,  $L$  has a principal eigenvalue.*

*Proof.* Without loss of generality, assume  $x^* = 0$ . From [5], we can get a lower bound on the spectral radius by observing that:

$$A_\lambda u \geq cu \implies \text{spr}(A_\lambda) \geq c \quad (3.3)$$

We can then explicitly borrow the test function from [7]. Fix  $\delta, \varepsilon$  such that  $J(x, y) \geq \varepsilon$  for all  $|x|, |y| \leq \delta$ . For every  $\gamma > 0$  write  $u_\gamma$ :

$$u_\gamma(x) = \begin{cases} \frac{1}{\gamma + a(x) - a(0)} & \text{if } |x| < \delta \\ 0 & \text{else} \end{cases} \quad (3.4)$$

We have the inequalities:

$$\begin{aligned} Ju_\gamma(x) &= \int_{-\delta}^{\delta} \frac{J(x, y)}{\gamma + a(y) - a(0)} dy \\ &\geq \varepsilon \int_{-\delta}^{\delta} \frac{1}{\gamma + C|y|} dy \geq \frac{\varepsilon}{C} \ln \left[ \frac{C\delta + \gamma}{\gamma} \right] \end{aligned} \quad (3.5)$$

Choose  $\gamma > 0$  such that the last term above is greater than one and  $-a(0) < \lambda < \gamma - a(0)$ . Using the above inequality gives:

$$\begin{aligned} A_\lambda u_\gamma(x) &= \frac{J}{\lambda + a(x)} \\ &\geq \frac{1}{\gamma + a(x) - a(0)} \frac{\varepsilon}{C} \ln \left[ \frac{C\delta + \gamma}{\gamma} \right] \\ &\geq u_\gamma(x) \end{aligned} \quad (3.6)$$

We have now shown that  $\text{spr}(A_\lambda) \geq 1$  and can invoke Theorem 1.  $\square$



**Proposition 5.** *Assume that  $a(x)$  converges to its infimum at either  $\pm\infty$ , and there exists  $c, c'$  such that  $J(x, y) > \varepsilon$  for  $|y - x| < \delta$  for all  $x \in \Omega$ . Then,  $L$  has a principal eigenvalue.*

*Proof.* Without loss of generality, assume  $a(x) \rightarrow \inf a$  as  $x \rightarrow \infty$ . We can construct almost the same inequality as in the proof of Proposition 4. Choose  $\lambda, z$  such that:

$$\frac{\varepsilon}{\lambda + a(x)} > 1 \quad \text{for } |x - z| < \delta \quad (3.7)$$

Choosing  $u = \mathbf{1}_{(z-\delta, z+\delta)}$  combined with the inequality in [5] completes the proof.  $\square$

Note that the conditions on  $J$  for Proposition 5 are easily satisfied if  $J(x, y) = f(x - y)$  and  $f > c > 0$  on some neighborhood of zero. The proof of that proposition could also be easily generalized. Without enumerating all possible cases, it is obvious that other conditions on  $J$  can be used for other similar cases, e.g.  $a(x) = \frac{1}{1+|x|} + \cos(\theta)$  and that weaker conditions on  $J$  can be found for the cases listed above.

**Proposition 6.** *Assume  $a(x)$  is locally Lipschitz and reaches its infimum at  $x^*$  and  $J(x, y) = J(y, x)$ . Also, assume the range of  $J$  is in  $L^2$  and there exists  $\delta, \varepsilon > 0$  and a bounded set  $U$  such that:*

$$\int_U J(x, y) dx > \varepsilon \quad \text{for } |y - x^*| < \delta \text{ a.e.} \quad (3.8)$$

*Then,  $L$  has a principal eigenvalue.*

*Proof.* Again, we will assume  $x^* = 0$  for the sake of notation. Define the family of inner-products  $\langle \cdot, \cdot \rangle_{\lambda+a}$  by:

$$\langle f, g \rangle_{\lambda+a} = \int_{\Omega} f(x) \bar{g}(x) (\lambda + a(x)) dx \quad (3.9)$$

It is easy to observe that  $A_{\lambda}$  is self-adjoint with respect to the inner-product  $\langle \cdot, \cdot \rangle_{\lambda+a}$  and that  $\langle \cdot, \cdot \rangle_{\lambda+a}$  is topologically equivalent to the usual  $L^2$  inner-product for all  $\lambda > -\inf a$ .

Need to show there exists  $v$  and  $\lambda$  such that:

$$\frac{\langle A_{\lambda} v, v \rangle_{\lambda+a}}{\langle v, v \rangle_{\lambda+a}} > 1 \quad (3.10)$$

Use:

$$v_\lambda = \frac{\mathbf{1}_{(-\delta, \delta)}}{\lambda + a(x)} + \mathbf{1}_U \quad (3.11)$$

where  $U$  and  $\delta$  are as in the statement of the proposition. Choose some  $\lambda^* > -\inf a$  and observe that:

$$\langle v_\lambda, v_\lambda \rangle_{\lambda+a} \leq \langle v_{\lambda^*}, v_{\lambda^*} \rangle_{\lambda^*+a} = C \quad (3.12)$$

for all  $\lambda \leq \lambda^*$ .

Define  $\gamma = \lambda - \inf a$ . We can now construct the sequence of inequalities:

$$\begin{aligned} \langle A_\lambda v_\lambda, v_\lambda \rangle_{\lambda+a} &= \left\langle \frac{Jv_\lambda}{\lambda + a}, v_\lambda \right\rangle_{\lambda+a} \\ &= \langle Jv_\lambda, v_\lambda \rangle \\ &\geq \int_\Omega \frac{J\mathbf{1}_U(x)}{\lambda + a(x)} dx \\ &\geq \int_{-\delta}^{\delta} \frac{\varepsilon}{\gamma + c|x|} dx \geq \frac{\varepsilon}{c} \ln \left[ \frac{c\delta + \gamma}{\gamma} \right] \end{aligned} \quad (3.13)$$

Choosing  $\lambda$  sufficiently small gives  $\langle A_\lambda v_\lambda, v_\lambda \rangle_{\lambda+a} > C$ , which gives the desired result.  $\square$

Finally, a counter-example to show that the Lipschitz hypothesis is necessary when  $a(x)$  reaches its global minimum and is bounded away from that minimum elsewhere. For simplicity, we will take  $\Omega = S^1$ . Take the family of systems where  $J = \frac{1}{2\pi}$  and  $a(x) = c|x|^\alpha + c'$  where  $c, c' > 0$  and  $0 < \alpha < 1$ . Obviously,  $a$  is  $\alpha$ -Hölder continuous. Assume we have an eigenpair  $\mu, v(x)$ . Dividing through wherever  $a + \mu \neq 0$ , we have:

$$v = \frac{\int v(\omega) d\omega}{c|\theta|^\alpha + c' + \mu} \quad (3.14)$$

That means  $\int v(\omega) d\omega = 1$  and  $v = \frac{1}{c|\theta|^\alpha + c' + \mu}$ . However, if we set:

$$c = 2 \int_{S^1} |\theta|^{-\alpha} d\theta \quad (3.15)$$

the above equations are not solvable for any pair  $c, \mu$  since  $\int v(\omega) d\omega \leq \frac{1}{2}$  for all  $\mu \geq -c'$ . If  $\mu < -c'$ , the integral of the potential eigenfunction is no longer defined. That implies that there is not any eigenvalues for  $A$ .

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